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Discretization of Nonlinear Systems with Delayed Multi-Input via Taylor Series and Scaling and Squaring Technique

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An input time delay always exists in practical systems. Analysis of the delay phenomenon in a continuous-time domain is sophisticated. It is appropriate to obtain its corresponding discrete-time model for implementation via digital computers. In this paper a new scheme for the discretization of nonlinear systems using Taylor series expansion and the zero-order hold assumption is proposed. The mathematical structure of the new discretization method is analyzed. On the basis of this structure the sampled-data representation of nonlinear systems with time-delayed multi-input is presented. The delayed multi-input general equation has been derived. In particular, the effect of the time-discretization method on key properties of nonlinear control systems, such as equilibrium properties and asymptotic stability, is examined. Additionally, hybrid discretization schemes that result from a combination of the scaling and squaring technique (SST) with the Taylor series expansion are also proposed, especially under conditions of very low sampling rates. Practical issues associated with the selection of the method's parameters to meet CPU time and accuracy requirements, are examined as well. A performance of the proposed method is evaluated using a nonlinear system with time delay: maneuvering an automobile.

Key Words: Multi-input, Nonlinear System, Scaling and Squaring Technique, Stability, Taylor-series, Time-delay, Time-discretization

1. Introduction

Time-delay systems (shortly, TDS) are also referred to as systems with aftereffect or dead-time, hereditary systems, equations with a deviating argument or differential-difference equations. The future of internet technology involves the developing and evolving of results in the interest of control systems with time delay. The convergence of communication and computation in control systems and the complex behavior of the control systems with non-negligible time-delays are the two main reasons for the special attention in the time-delayed status. The digital controller using the communication and the increased computation requirement in the system induces the time-delay. In the embedded control systems, the effect of the time-delay due to the communication and the increased computation cannot be ignored. Also, control systems with time-delays exhibit complex behavior because of their infinite dimensionality. Even in the case of linear time-invariant systems that have constant time-delays in the input or states have infinite dimensionality if expressed in the continuous time domain. For this reason, it is difficult to apply the controller design technique developed during the last several years for finite-dimensional systems to the systems with any time-delays in the variables. Thus, control system design methods that can assist the system with time-delays are necessary.

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Time delay is often encountered in various engineering systems, such as chemical processes, hydraulic, and rolling mill systems and its existence is frequently a source of instability. Many of these models are also significantly nonlinear which motivates research in the control of nonlinear systems with time delay. A natural direction is to try to extend the ideas and results of nonlinear non-delay control to systems with delay. Such results include the input-output linearization and decoupling, partial feedback linearization with delay term domination, and extension of control Lyapunov functions (CLF) to delay systems in the form of control Lyapunov-Razumikhin functions (CLRF). Jankovic (2003) established a generic sufficient condition for stabilization using the domination redesign formula. Su and Huang (1992) presented a delay-dependent stability condition for the linear uncertain time-delay systems. Wu (1996) derived some robust stability conditions which lead to some bounds on the perturbations so that the LQG optimal control system can remain stable in the presence of delayed perturbations. In many applications magnetic levitation systems are required to have a large operating range. Choi and Baek (2002) applied Time Delay Control (TDC) to a single-axis magnetic levitation system to solve this problem. Lee and Kim (2003) proposed a high level CVT ratio control algorithm to improve the engine performance by considering the powertrain response time delay. Cho and Park (2004) proposed a new impedance controller for bilateral tele-operation under a time delay.

Traditional numerical schemes for ordinary differential equations, such as Runge-Kutta schemes, usually fail to attain their asserted order when applied to ordinary differential control equations due to the measurability of the control functions. Grune and Kloeden (2002) extend a systematic method for the derivation of high order schemes for affine controlled nonlinear systems to a larger class of systems in which the control variables are allowed to appear nonlinearly in multiplicative terms.

Hong and Wu (1994) derived sufficient conditions for the zeros of the polynomial to be either inside the unit disk in the complex plane or at least one zero outside the unit disk by examining the coefficients of a given polynomial in the linear discrete system. Kang and Park (1999) experimentally confirm the fundamental dynamic properties of an electro-dynamic structure. The discretization effects are examined for the conversion of continuous properties such as mass, stiffness, and surface charge into discrete quantities. In the systems considered, the linearized characteristics are well-matched with the nonlinear systems in the sense that the linearized effects predominate over the high-order nonlinear terms.

In the field of the discretization, for the original continuous-time systems in the time free case (Franklin et al,. 1998) the traditional numerical techniques such as the Euler and Runge-Kutta methods have been used to obtain the sampled-data representations. However, these methods require a small sampling time interval. This occurs because it is necessary to meet the desired accuracy and because they cannot be applied to the large sampling period case. However, due to the physical and technical limitation slow sampling is becoming inevitable. A time-discretization method which expands the well-known time-discretization of the linear time-delay system (Franklin et al., 1998; Vaccaro, 1995) to nonlinear continuous-time control system with time-delay (Kazantzis et al., 2003) can solve this problem. This method is applied to the nonlinear control systems with delayed multi-input (Park et al., 2004) and the nonlinear control systems with non-affine delayed input (Park et al., 2004). The effect of this approach on system-theoretic properties of nonlinear systems, such as equilibrium properties, relative order, stability, zero dynamics and minimum-phase characteristics, which all highlight the natural and transparent way in which Taylor methods permeate the relevant theoretical aspects, is also studied (Kazantzis et al., 1997).

Nowadays, modern nonlinear control strategies are usually implemented on a microcontroller or digital signal processor. As a direct consequence, the control algorithm has to work in discrete-time. For such digital control algorithms, one of the following time discretization approaches is typically used: time-discretization of a continuous time control law designed on the basis of a continuous time system; and time-discretization of a continuous time system resulting in a discrete-time system and control law design in discrete-time.

It is apparent that the second approach is an attractive feature for dealing directly with the issue of sampling. Indeed, the effect of sampling on system-theoretic properties of the continuous-time system is very important because they are associated with the attainment of the design objectives. It should be emphasized that in both design approaches time discretization of either the controller or the system model is necessary. Furthermore, notice that in the controller design for time-delay systems, the first approach is troublesome because of the infinite-dimensional nature of the underlying system dynamics. As a result the second approach becomes more desirable and will be pursued in the present study.

This paper proposed the time discretization method of the nonlinear control systems with multiple time-delays in the control (Park et al., 2004). The proposed discretization scheme applies the Taylor Series expansion according to the mathematical structure developed for the delay-free nonlinear system (Kazantzis and Kravaris, 1997; 1999) and delayed single-input nonlinear system (Kazantzis et al., 2003). The effect of sampling on system-theoretic properties of nonlinear systems with time-delayed multi-input, such as equilibrium properties and stability, is examined. Finally the well-known SST, which is widely used for computing the matrix exponential (Higham, 2004), is expanded to the nonlinear case when the sampling period is too large.

2. Nonlinear System with Delayed Input

In the present study single-input nonlinear continuous-time control systems are considered with a state-space representation of the form (Kazantzis et al., 2003):

$$\frac{dx(t)}{dt}=f(x(t))+g(x(t))u(t-D) \qquad (1)$$

where $x \in X \subset R^n$ is the vector of states and X is an open and connected set, $u \in R$ is the input variable and D is the system's constant time-delay (dead-time) that directly affects the input. It is assumed that $f(x)$, $g(x)$ are real analytic vector fields on X.

An equidistant grid on the time axis with mesh $T=t_{k+1}-t_k>0$ is considered, where $[t_k, t_{k+1})=[kT, (k+1)T)$ is the sampling interval and T is the sampling period. It is assumed that system (1) is driven by an input that is piecewise constant over the sampling interval, i.e. the zero-order hold (ZOH) assumption holds true:

$$u(t)=u(kT)\equiv u(k)=constant \qquad (2)$$

for $kT \le t < kT+T$. Furthermore, let:

$$D=qT+\gamma \qquad (3)$$

where $q \in \{0, 1, 2, \cdots\}$ and, $0<\gamma<T$ i.e. the time-delay D can be represented as an integer multiple of the sampling period plus a fractional part of T (Franklin et al., 1998; Vaccaro, 1995). Based on the ZOH assumption and the above notation one can deduce that the delayed input variable attains the following two distinct values within the sampling interval (Vaccaro, 1995):

$$u(t-D)=\begin{cases} u(kT-qT-T)\equiv u(k-q-1) & \text{if } kT \le t \le kT+\gamma \\ u(kT-qT)\equiv u(k-q) & \text{if } kT+\gamma \le t < kT+T \end{cases} \qquad (4)$$

Under the above preliminaries, the time-discretization of nonlinear systems with delay-free single input and time-delay single-input will be discussed briefly.

2.1 Discretization of nonlinear system with Delay-Free input

Initially, delay-free ($D=0$) nonlinear control systems are considered with a state-space representation of the form:

$$\frac{dx(t)}{dt}=f(x(t))+u(t)g(x(t)) \qquad (5)$$

Within the sampling interval and under the ZOH assumption, the solution of (5) is expanded in a uniformly convergent Taylor series (Vydyasagar, 1978):

$$\begin{aligned} x(k+1) &= \Phi_T(x(k),\ u(k)) \\ &= x(k) + \sum_{l=1}^{\infty} \frac{T^l d^l x}{l!\ dt^l}\Big|_{tk} \\ &= x(k) + \sum_{l=1}^{\infty} A^{[l]}(x(k),\ u(k)) \frac{T^l}{l!} \end{aligned} \tag{6}$$

where $x(k)$ is the value of the state vectorat time $t=t_k=kT$ and $A^{[l]}(x,\ u)$ are determined recursively by:

$$A^{[1]}(x,\ u) = f(x) + ug(x)$$

$$A^{[l+1]}(x,\ u) = \frac{\partial A^{[l]}(x,\ u)}{\partial x}(f(x) + ug(x)) \tag{7}$$

$$\text{with } l=1,\ 2,\ 3,\ \cdots$$

Remark 1: In general $A^{[l]}(x,\ u)$ is a l-th degree polynomial in the input variable u (Kazantzis et al., 2003):

$$\begin{aligned} A^{[l]}(x,\ u) &= a_0^{[l]}(x) + a_1^{[l]}(x)u + a_2^{[l]}(x)u^2 \\ &\quad + \cdots + a_l^{[l]}(x)u^l \end{aligned} \tag{8}$$

In view of the representation (8), the series expansion (6) can be rewritten as follows:

$$\begin{aligned} x(k+1) &= \Phi_T(x(k),\ u(k)) \\ &= x(k) + \sum_{l=1}^{\infty}\sum_{m=1}^{l} [u(k)]^m a_m^{[l]}(x(k)) \frac{T^l}{l!} \end{aligned} \tag{9}$$

It should noted, that the series expansion (6) can also be expressed in an operator form. Indeed, under the ZOH assumption, the new discretization approach can be naturally formulated within the context of Lie series theory for nonlinear autonomous ODEs. The following definition is deemed essential.

Definition 1 Given f, an analytic vector field on R^n and h, an analytic scalar field on R^n, the Lie derivative of h with respect to f is defined in local coordinates as (Kazantzis et al., 2003):

$$L_f h(x) = \frac{\partial h}{\partial x_1} f_1 + \cdots + \frac{\partial h}{\partial x_n} f_n \tag{10}$$

In light of Definition 1, the solution to the recursive relation (7) may be represented in terms of higher-order Lie derivatives as follows:

$$A_i^{[l]}(x,\ u) = (L_f + uL_g)^l x_i \tag{11}$$

where the subscript $i=1,\ \cdots,\ n$ denotes the i-th component and $L_f = \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}$, $L_g = \sum_{i=1}^{n} g_i(x)\frac{\partial}{\partial x_i}$ are Lie derivative operators. This allows for representing the series expansion (6) as a uniformly convergent Lie series for the ESDR (exact sampled-data representation):

$$\begin{aligned} x_i(k+1) &= \Phi_{i,T}(x(k),\ u(k)) \\ &= x_i(k) + \sum_{l=1}^{\infty} (L_f + uL_g)^l x_i\big|_{(x(k),\ u(k))} \frac{T^l}{l!} \end{aligned} \tag{12}$$

and similarly for the ASDR (approximate sampled-data representation):

$$\begin{aligned} x_i(k+1) &= \Phi_{i,T}^N(x(k),\ u(k)) \\ &= x_i(k) + \sum_{l=1}^{N} (L_f + uL_g)^l x_i\big|_{(x(k),\ u(k))} \frac{T^l}{l!} \end{aligned} \tag{13}$$

with $i=1,\ \cdots,\ n$.

2.2 Discretization of nonlinear system with delayed single-input

The sampled-data representation of the nonlinear system with single delayed input can be derived from Eq. (6), which provides the following Eq. (14) (Kazantzis et al., 2003).

$$\begin{aligned} &x(kT+\gamma) = x(kT) + \sum_{l=1}^{\infty} A^l(x(kT),\ u(k-q-1))\frac{\gamma^l}{l!} \\ &\qquad \text{if } kT \le T < kT+\gamma \\ &x(kT+T) = x(kT+\gamma) + \sum_{l=1}^{\infty} A^l(x(kT+\gamma),\ u(k-q))\frac{(T-\gamma)^l}{l!} \\ &\qquad \text{if } kT+\gamma \le t < kT+T \end{aligned} \tag{14}$$

where $x(k)$ and $A^{[l]}$ are the same as above delay-free case.

3. General equation derivation

3.1 Multi-input linear system with time delay

General equation derivation of multi-input linear system in state space form with time delay can be represented as follows.

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + \sum_{i=1}^{n} b_i u_i(t-D_i) \\ &= Ax(t) + b_1 u_1(t-D_1) + b_2 u_2(t-D_2) \\ &\quad + \cdots + b_n u_n(t-D_n) \end{aligned} \tag{15}$$

where A, $b_i(i=1, \cdots, n)$ are constant matrices of appropriate dimensions. Under the ZOH assumption interpreting (15) within any time interval $I=[t_a, t_b)$, where u_i=constant $(i=1, \cdots, n)$ results in :

$$x(t_b)=\exp(A(t_b-t_a))x(t_a) + \int_{t_a}^{t_b}\exp(A(t_b-\tau))(b_1u_1+b_2u_2+\cdots+b_nu_n)d\tau \quad (16)$$

Assuming $D_i=q_i+\gamma_i$, $I=[t_i, t_f)$
where q_i and γ_i have the same meaning as the q and γ of part 2.

It is convenient to assume that.

Case 1. $kT\leq t<kT+\gamma_1$

$u_1(t-D_1)=u_1(k-q_1-1)$

$u_2(t-D_2)=u_2(k-q_2-1)$

$\cdots$

$u_n(t-D_n)=u_n(k-q_n-1)$

$$x(kT+\gamma_1)=\exp(A\gamma_1)x(kT) + \int_{kT}^{kT+\gamma_1}\exp(A(kT+\gamma_1-\tau))(b_1u_1(k-q_1-1) + b_2u_2(k-q_2-1)+\cdots+b_nu_n(k-q_n))d\tau \quad (17)$$

And the exponential matrix is defined through the uniformly convergent power series :

$$\exp(At)=\sum_{t=0}^{\infty}\frac{A^lt^l}{l!} \quad (18)$$

$$\Rightarrow x(kT+\gamma_1) =x(k)+\sum_{l=1}^{\infty}[A^{l-1}(Ak(k)+b_1u_1(k-q_1-1) + b_2u_2(k-q_2-1)+\cdots+b_nu_n(k-q_n-1))]\frac{\gamma_1^l}{l!} \quad (19)$$

$\cdots$

Case 2. $kT+\gamma_m\leq t<kT+\gamma_{m+1}$

$u_1(t-D_1)=u_1(k-q_1)$

$\cdots$

$u_m(t-D_m)=u_m(k-q_m)$

$u_{m+1}(t-D_{m+1})=u_{m+1}(k-q_{m+1}-1)$

$\cdots$

$u_n(t-D_n)=u_n(k-q_n-1)$

$$x(kT+\gamma_{m+1}) =x(kT+\gamma_m)+\sum_{l=1}^{\infty}[A^{l-1}(Ax(kT+\gamma_m)+b_1u_1(k-q_1) +\cdots+b_mu_m(k-q_m)+b_{m+1}u_{m+1}(k-q_{m+1}-1) +\cdots+b_nu_n(k-q_n-1))]\frac{(\gamma_{m+1}-\gamma_m)^l}{l!} \quad (20)$$

$\cdots$

Case 3. $kT+\gamma_n\leq t<kT+T$

$u_1(t-D_1)=u_1(k-q_1)$

$u_2(t-D_2)=u_2(k-q_2)$

$\cdots$

$u_n(t-D_n)=u_n(k-q_n)$

$$x(kT+T) =x(kT+\gamma_n)+\sum_{l=1}^{\infty}[A^{l-1}(Ax(kT+\gamma_n)+b_1u_1(k-q_1) +b_2u_2(k-q_2)+\cdots+b_nu_n(k-q_n))]\frac{(T-\gamma_n)^l}{l!} \quad (21)$$

3.2 Multi-input nonlinear system with time delay

The discretization scheme of nonlinear control systems with delayed two-input and three-input are presented in (Park and Chong, 2004).

The general multi-input nonlinear system in state space form with time delay can be represented as follows.

$$\begin{aligned}\frac{dx(t)}{dt}&=f(x(t))+\sum_{i=1}^{n}g_i(x(t))u_i(t-D_i)\\ &=f(x(t))+u_1(t-D_1)g_1(x(t))\\ &\quad+u_2(t-D_2)g_2(x(t))\\ &\quad+\cdots+u_n(t)g_n(x(t-D_n))\end{aligned} \quad (22)$$

The general time-discretization equation of nonlinear system with multi-input time-delay can be derived as follows :

i) $kT\leq t<kT+\gamma_1$

$$x(kT+\gamma_1) =x(kT)+\sum_{l=1}^{\infty}A^l(x(kT), u_1(k-q_1-1), \cdots, u_n(k-q_n-1))\frac{\gamma_1^l}{l!} \quad (23)$$

$\vdots$

ii) $kT+\gamma_i\leq t<kT+\gamma_{i+1}$ where $1\leq l\leq n-1$

$$x(kT+\gamma_{i+1}) =x(kT+\gamma_i)+\sum_{l=1}^{\infty}A^l(x(kT)+\gamma_i), u_1(k-q_1), \cdots, u_i(k-q_i), u_{i+1}(k-q_{i+1}-1), \cdots, u_n(k-q_n-1))\frac{(\gamma_{i+1}-\gamma_i)^l}{l!} \quad (24)$$

$\vdots$

iii) $kT+\gamma_n \le t < kT+T$

$$x(kT+T) = x(kT+\gamma_n)+\sum_{l=1}^{\infty} A^l(x(kT+\gamma_n), u_1(k-q_1), \cdots, u_{n-1}(k-q_{n-1}), (k-q_n))\frac{(T-\gamma_n)^l}{l!} \tag{25}$$

Remark 2: It is important to observe, that the ESDR of Eqs. (23)~(25) represents the nonlinear analogue of the exact discretization scheme available for linear systems Eqs. (19)~(21).

Theorem 1. Let x^0 be an equilibrium point of the original nonlinear continuous-time system: $\frac{dx(t)}{dt}=f(x)+u_1g_1(x)+u_2g_2(x)+\cdots+u_ng_n(x)$, belongs to the equilibrium manifold: $E^c=\{x\in R^n \mid \exists u_i(i=1, 2, \cdots, n)\in R: f(x)+u_1g_1(x)+u_2g_2(x)+\cdots+u_ng_n(x)=0\}$, and $u_i=u_i^0(i=1, 2, \cdots, n)$ be the corresponding equilibrium value of the input variables: $f(x^0)+u_1^0g_1(x^0)+u_2^0g_2(x^0)+\cdots+u_n^0g_n(x^0)=0$. The x^0 belongs to the equilibrium manifold: $E^c=\{x\in R^n \mid \exists u_i(i=1, 2, \cdots, n)\in R: \Phi_T^D(x, u_1, u_2, \cdots, u_n)=x\}$ of the ESDR: $x(k+1)=\Phi_T^D(x(k), u_1(x-q_1-1), u_1(k-q_1), \cdots, u_n(k-q_n-1), u_n(k-q_n))$ and ASDR: $x(k+1)=\Phi_T^{N,D}(x(k), u_1(k-q_1-1), u_1(k-q_1), \cdots, u_n(k-q_n-1), u_n(k-q_n))$ obtained under the proposed Taylor-Lie discretization method, with $u_i=u_i^0(i=1, 2, \cdots, n)$ being the corresponding equilibrium values of the input variables: $\Phi_T^D(x^0, u_1^0, u_2^0, \cdots, u_n^0)=x^0$ and $\Phi_T^{N,D}(x^0, u_1^0, u_2^0, \cdots, u_n^0)=x^0$.

Proof: x^0 represents the equilibrium point and $u_i^0(i=1, 2, \cdots, n)$ are the corresponding equilibrium values of the input variables.

$$\Rightarrow A^{[1]}(x^0, u_1^0, u_2^0, \cdots, u_n^0) = f(x^0)+u_1^0g_1(x^0)+u_2^0g_2(x^0)+\cdots+u_n^0g_n(x^0)=0$$

$$\Rightarrow A^{[l+1]}(x^0, u_1^0, u_2^0, \cdots, u_n^0) = \frac{\partial A^{[l]}(x^0, u_1^0, u_2^0, \cdots, u_n^0)}{\partial x}A^{[1]}(x^0, u_1^0, u_2^0, \cdots, u_n^0)=0$$

for all $l\in\{1, 2, 3, \cdots\}$

$$\Rightarrow \Phi_{\gamma_1}(x^0, u_1^0, u_2^0, \cdots, u_n^0) = x^0+\sum_{l=1}^{\infty}A^{[l]}(x^0, u_1^0, u_2^0, \cdots, u_n^0)\frac{\gamma_1^l}{l!}=x^0$$

$$\Rightarrow \Phi_{\gamma_2-\gamma_1}(\Phi_{\gamma_1}(x^0, u_1^0, u_2^0, \cdots, u_n^0), u_1^0, u_2^0, \cdots, u_n^0)=x^0$$

$\cdots$

$$\Phi_T^D(x^0, u_1^0, u_2^0, u_n^0) = \Phi_{T-\gamma_n}(\Phi_{\gamma_n-\gamma_{n-1}}(\cdots(\Phi_{\gamma_2-\gamma_1}(\Phi_{\gamma_1}(x^0, u_1^0, u_2^0, \cdots, u_n^0), u_1^0, u_2^0, \cdots, u_n^0))\cdots))=x^0$$

Similar arguments apply to the $\Phi_T^{N,D}$ map of the ASDR. Therefore, x^0 belongs to the equilibrium manifold E^d of the ESDR and ASDR for any finite truncation order N.

Theorem 1 essentially states that equilibrium properties are preserved under the proposed Taylor discretization method.

Theorem 2. Assume that matrix $M=\left[\frac{\partial f}{\partial x}+u_1^0\frac{\partial g_1}{\partial x}+u_2^0\frac{\partial g_2}{\partial x}+\cdots+u_n^0\frac{\partial g_n}{\partial x}\right](x^0)$ is Hurwitz, so that x^0 is a locally asymptotically stable equilibrium point of the delay-free system:

$$\frac{dx(t)}{dt}=f(x(t))+g_1(x(t))u_1(t) +g_2(x(t))u_2(t)+\cdots+g_n(x(t))u_n(t)$$

Then:

(1) x^0 is a locally asymptotically stable equilibrium point of the ESDR.

(2) x^0 is a locally asymptotically stable equilibrium point of the ASDR for sufficiently large N, when T is fixed.

Proof: The following technical lemma is essential and its proof can be found in Kazantzis and Kravaris (1997).

Lemma 1.
In the single input status, let x^0 be an equilibrium point of $\frac{dx(x)}{dt}=f(x(t))+g(x(t))u(t-D)$ that corresponds to $u=u^0$. For any analytic scalar field $h(x)$ and positive integer l the following equality holds:

$$\frac{\partial}{\partial x}[L_f+uL_g]^l h(x)|_{(x^0,u^0)}=\frac{\partial h}{\partial x}\left[\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right]^l(x^0)$$

The i-th row of the matrix $\frac{\partial \Phi_T^N}{\partial x}(x^0, u^0)$ can be calculated as follows:

$$\frac{\partial \Phi_{i,T}^N}{\partial x}(x^0, u^0)=\sum_{l=0}^{N}\frac{\partial}{\partial x}[(L_f+uL_g)^l x_i]|_{(x^0,u^0)}\frac{T^l}{l!} =\sum_{l=0}^{N}\frac{\partial x_i}{\partial x}\left(\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right)^l(x^0)\frac{T^l}{l!}$$

Due to Lemma 1.

In the case of multi-input under the assumption of ZOH it is obvious that in any time interval we denote that u_i=constant $(i=1, 2, \cdots, n)$. So it is correct and convenient to regard the multi-input as a single input $g^*(x)u^*$.

Therefore:

$$\frac{\partial \Phi_T^N}{\partial x}(x^0, u^{*0}) = \sum_{l=0}^{N}\left(\frac{\partial f}{\partial x}+u^{*0}\frac{\partial g^*}{\partial x}\right)^l (x^0)\frac{T^l}{l!} = \sum_{l=0}^{N} M^l \frac{T^l}{l!}$$

for an ASDR of finite truncation order N, or

$$\frac{\partial \Phi_T}{\partial x}(x^0, u^{*0}) = \exp\left[\left(\frac{\partial f}{\partial x}+u^{*0}\frac{\partial g^*}{\partial x}\right)(x^0)\, T\right] = \exp(MT)$$

for the ESDR $(N\to\infty)$

(1) Consider now the ESDR with time-delay D. Notice that:

$$\begin{aligned}\frac{\partial \Phi_T^D}{\partial x}(x^0, u^{*0}) &= \frac{\partial \Phi_{T-\gamma_n}}{\partial x}(\Phi_{\gamma_n-\gamma_{n-1}}(\cdots(\Phi_{\gamma_1}(x^0, u^{*0}))\cdots), u^{*0}) \\ &\quad \frac{\partial \Phi_{\gamma_n-\gamma_{n-1}}}{\partial x}(\Phi_{\gamma_{n-1}-\gamma_{n-2}}(\cdots(\Phi_{\gamma_1}(x^0, u^{*0}))\cdots), u^{*0}) \\ &\quad \cdots \frac{\partial \Phi_{\gamma_1}}{\partial x}(x^0, u^{*0}) \\ &= \frac{\partial \Phi_{T-\gamma_n}}{\partial x}(x^0, u^{*0})\frac{\partial \Phi_{\gamma_n-\gamma_{n-1}}}{\partial x}(x^0, u^{*0})\cdots\frac{\partial \Phi_{\gamma_1}}{\partial x}(x^0, u^{*0}) \\ &= \exp(M(T-\gamma_n))\exp(M(\gamma_n-\gamma_{n-1}))\cdots\exp(M\gamma_1) \\ &= \exp(MT)\end{aligned}$$

Since M is Hurwitz, it can be inferred that all the eigenvalues of $\frac{\partial \Phi_T^D}{\partial x}(x^0, u^{*0})$ have modulus less than one, and hence x^0 is a locally asymptotically stable equilibrium point of the ESDR.

(2) Consider now the ASDR. One obtains:

$$\begin{aligned}\frac{\partial \Phi_T^{N,D}}{\partial x}(x^0, u^{*0}) &= \frac{\partial \Phi_{T-\gamma_n}}{\partial x}(\Phi_{\gamma_n-\gamma_{n-1}}(\cdots(\Phi_{\gamma_1}(x^0, u^{*0}))\cdots), u^{*0}) \\ &\quad \frac{\partial \Phi_{\gamma_n-\gamma_{n-1}}}{\partial x}(\Phi_{\gamma_{n-1}-\gamma_{n-2}}(\cdots(\Phi_{\gamma_1}(x^0, u^{*0}))\cdots), u^{*0}) \\ &\quad \cdots \frac{\partial \Phi_{\gamma_1}}{\partial x}(x^0, u^{*0}) \\ &= \frac{\partial \Phi_{T-\gamma_n}}{\partial x}(x^0, u^{*0})\frac{\partial \Phi_{\gamma_n-\gamma_{n-1}}}{\partial x}(x^0, u^{*0})\cdots\frac{\partial \Phi_{\gamma_1}}{\partial x}(x^0, u^{*0}) \\ &= \left\{\sum_{l_n=0}^{N} M^{l_n}\frac{(T-\gamma_n)^{l_n}}{l_n!}\right\}\left\{\sum_{l_{n-1}=0}^{N} M^{l_{n-1}}\frac{(\gamma_n-\gamma_{n-1})^{l_{n-1}}}{l_{n-l}!}\right\} \\ &\quad \cdots\left\{\sum_{l_n=0}^{N} M^{l_1}\frac{(\gamma_1)^{l_1}}{l_1!}\right\} \\ &= \sum_{l_n=0}^{N}\sum_{l_{n-1}=0}^{N}\cdots\sum_{l_n=0}^{N} M^{l_m+l_{m-1}+\cdots+l_1}\frac{(T-\gamma_n)^{l_n}(\gamma_n-\gamma_{n-1})^{l_{n-1}}\cdots(\gamma_1)^{l_1}}{l_n!\,l_{n-1}!\cdots l_1!}\end{aligned}$$

Notice now that for a stable eigenvalue λ_i of M $(\mathrm{Re}[\lambda_i]<0)$, the corresponding eigenvalue a_i of $\frac{\partial \Phi_T^{N,D}}{\partial x}(x^0, u^{*0})$:

$$a_i = \sum_{l_n=0}^{N}\sum_{l_{n-1}=0}^{N}\cdots\sum_{l_1=0}^{N}\lambda^{l_m+l_{m-1}+\cdots+l_1}\frac{(T-\gamma_n)^{l_n}(\gamma_n-\gamma_{n-1})^{l}{}_{n-1}\cdots(\gamma_1)^{l_1}}{l_n!\,l_{n-1}!\cdots l_1!}$$

is stable only when $|a_i|<1$.

Since for a fixed T and as $N\to\infty$, $a_i\to\exp(\lambda_i(\gamma_n-\gamma_{n-1}))\cdots\exp(\lambda_i\gamma_1)=\exp(\lambda_i T)$ one can always find a sufficiently large order of truncation N such that: $|a_i|<1$.

4. Scaling and Squaring Technique (SST)

The Taylor series expansion method can be applied to discretize the nonlinear systems with delayed input and provide the desired accurate result. In the case of a small sampling period, a small Taylor order N can satisfy the accuracy requirement. However, when the sampling interval T is large the order N of the Taylor method should also be large, in order for the necessary accuracy to be achieved. This is mathematically reflected upon the asymptotic behavior of $\frac{T^n}{(N+1)!}\to 0$ as $N\to\infty$. When T is considerably large $\frac{A^{[l]}T^l}{l!}$ might become extremely large due to the finite-precision arithmetic before it becomes small at higher powers, where convergence guarantees it. In the case of linear system this phenomenon occurs when calculating e^{AT} and $\int_0^T e^{AT}dt$, which causes overflow of the computer number representation.

The SST, which is also referred to as extrapolation to the limit technique in most numerical analysis literature, can be applied to solve this kind of problem. This technique is popularly used to calculate the exponential matrix $\exp(AT)$ for large sampling periods. By applying SST one can subdivide the sampling interval T into two or more subintervals of equal length. Actually an appropriate positive integer m can be chosen so that $T/2^m$ is small enough to calculate the exponential matrix. In this case the sampling period T is subdivided into 2^m equally spaced subintervals of length $T/2^m$ and the exponential matrix is calculated for a short interval $T/2^m$. Finally, the computation of $\exp(AT)$ is performed by squaring the matrix $\exp(AT/2^m)$ m times:

$$\exp(AT)=\left(\left(\left(\exp\left(A\frac{T}{2^m}\right)\right)^2\right)^{\cdots}\right)^2 \quad (26)$$

By applying the Taylor series expansion method the popular SST can be simply extended to the nonlinear case. After conducting some analogue one can use nonlinear operators and powers of operators to substitute matrices and matrix products. In detail the key idea for the nonlinear analogue of the SST remains the same as linear case.

In the nonlinear case when T is large enough one can divide the interval $[t_k, t_{+1})$ to 2^m equally spaced subintervals and use a small Taylor expansion order N with a time step $T/2^m$ for the 2^m intermediate subintervals to substitute the larger order N' used in the single-step Taylor method case. That is to assume now that $\Omega(N', T)$: $R^n \rightarrow R^n$ is the operator that corresponds to the Taylor expansion of order N' with a time step T, and that when it acts on $x(kT)$ the outcome is:

$$x(kT+T)=\Omega(N', T)x(kT) \quad (27)$$

where

$$\Omega(N', T)(\cdot)=I+\sum_{l=1}^{N'}A^{[l]}(x(k), u(k))\frac{T^l}{l!} \quad (28)$$

Using operator notation the resulting discrete-time system may now be written as follows:

$$x(kT+T)=\left[\Omega\left(N, \frac{T}{2^m}\right)\right]^{2m}x(kT) \quad (29)$$

The above ASDR may be viewed as the direct result of the combination of Taylor's method and the SST.

The choice of the parameters of N and m is important for implementation. Different values of N and m can reflect different requirements of the discretization performance. In this paper we use the following two factors: i) simplicity and computing time; and ii) numerical convergence and accuracy requirements; to select these two kinds of parameters. In fact the criterion of selecting an appropriate m is to compare the magnitude of the sampling period T with the fastest time constant $1/\rho$ of the original continuous-time system. If T is small compared to $2/\rho$, that is the sampling period is so small that only a single step Taylor method with a small Taylor order N can be applied to achieve superior results, one can set $m=0$. When T is larger than the fastest time constant $1/\rho$, one should apply the SST to reduce the Taylor order N. The sampling interval is therefore subdivided into 2^m subintervals, and a low-order N single step Taylor discretization method is applied for each subinterval. Consequently, it requires that the following inequality should hold:

$$\frac{T}{2^m}<\frac{2}{\rho} \quad (30)$$

Since the requirements for numerical convergence and stability are also met, the positive integer m is now selected to be:

$$m=\max\left(\left[\log_2\left(\frac{T}{\theta}\right)\right]+1, 0\right) \quad (31)$$

where $\theta<2/\rho$ is arbitrarily chosen and $[x]$ denotes the integer part of the number x. It is evident, that smaller values of the arbitrarily selected number θ would result to more stringent bounds on $T/2^m$.

The SST can also be applied to the nonlinear control systems with delayed multi-input. In this case we do not consider the single sampling interval T but the subintervals of γ_1, $\gamma_2-\gamma_1$, $\cdots$, $T-\gamma_n$ in one sampling period. The method applied to choose m can also be used by changing T of that preceding equality into these subintervals of γ_1, $\gamma_2-\gamma_1$, $\cdots T-\gamma_n$.

That is, $m_{\gamma_1}=\max\left(\left[\log_2\left(\frac{\gamma_1}{\theta}\right)\right]+1, 0\right)$,

$m_{\gamma_2-\gamma_1}=\max\left(\left[\log_2\left(\frac{\gamma_2-\gamma_1}{\theta}\right)\right]+1, 0\right)$, $\cdots$,

$m_{T-\gamma_n}=\max\left(\left[\log_2\left(\frac{T-\gamma_n}{\theta}\right)\right]+1, 0\right)$

5. Computer Simulations

A simplified model of maneuvering an automobile (Nijmeijer and Schaft, 1990) is considered in the computer simulations. Exact solutions for the systems are required in order to validate the proposed discretization method of nonlinear systems with the delayed multi-input. In this paper the continuous Matlab ODE solver is used as an exact solution. In the simulation the discrete values obtained using the Taylor series expansion method are compared with the values obtained through the continuous Matlab ODE solver at the corresponding sampled period. The Matlab ODE solver is accurate enough to be used as the exact solution in (Kazantzis et al., 2003). The partial derivative terms involved in the Taylor series expansion are determined recursively. For all the case studies considered these partial derivative terms are calculated using Maple.

The front axle of a simplified automobile maneuvering system is shown in Fig. 1. The middle of the axles linking the front wheels has position $(x_1, x_2) \in R^2$, while the rotation of this axis is given by the angle x_3. The states x_1, x_2 related with rolling are directly controlled by inputand the state x_3 related with rotation is directly controlled by u_2, thus the governing nonlinear differential equation can be obtained as follows;

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix} u_1(t-D_1) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2(t-D_2) \qquad (32)$$

The eigenvalues of the linear approximation of this system are small, thus $2/\rho$ becomes large. At first we choose a small sampling period and small time delay to verify the discretization method proposed in this paper. The inputs of u_1 and u_2 are assumed to be step functions respectively whose magnitudes are $u_1=1$ and $u_2=2.5$.

The initial conditions are $x_1(0)=0m$, $x_2(0)=0m$, $x_3(0)=30°$ and the sampling period (T) is 0.002 sec. The inputs delays are 0.0015 sec for u_1 and 0.0036 sec for u_2. Thus we can use a single-step Taylor method and choose $N=3$. As shown in Fig. 1, the numerical differences between the Matlab ODE solver and the proposed method for state x_1 range from 0.019×10^{-8} to -3.296×10^{-8}, and those values for state x_2 range from -4.636×10^{-8} to -5.823×10^{-8}. The differences between the responses of the Taylor method and the Matlab solver are presented in Fig. 2.

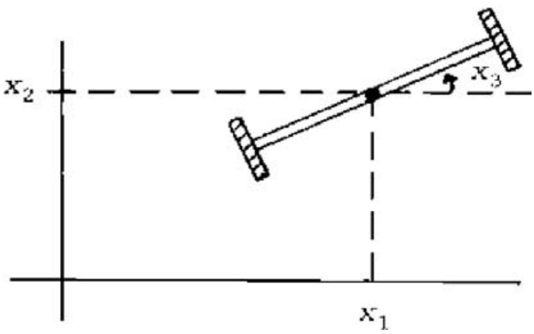

Fig. 1 Front axis of automobile

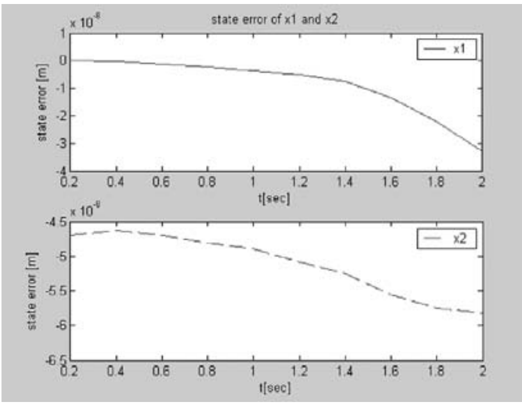

Fig. 2 State error responses of the simplified automobile for the case 1

Another test with $T=0.5$ sec, $D_1=0.3$ sec and $D_2=1.2$ sec was evaluated. It was conducted with a single-step Taylor method with $N=1, 3, 7$. The numerical differences between the Matlab ODE solver and the proposed method for state x_1 and x_2 are shown in Table 2. From Table 2 it is evident that if the sampling period T is enlarged, one must use larger N in order to achieve the desired accuracy. From the above definition we know that $q_1=0$, $\gamma_1=0.3$, $q_2=2$, $\gamma_2=0.2$. Assuming $l=m_{\gamma_2}$, $m=m_{\gamma_1-\gamma_2}$ and $n=m_{T-\gamma_1}$ are the scaling and squaring coefficients m of the intervals of $[kT, kT+\gamma_2)$, $[kT+\gamma_2, kT+\gamma_1)$ and

Table 1 The response of system in the simplest case

Time step	Matlab (x_1)	Taylor (x_1)	Matlab (x_2)	Taylor(x_2)
100	0.1363	0.1363	0.1415	0.1415
200	0.3250	0.3250	0.2012	0.2012
300	0.5192	0.5192	0.1631	0.1631
400	0.6714	0.6714	0.0365	0.0365
500	0.7442	0.7442	−0.1475	−0.1475
600	0.7200	0.7200	−0.3439	−0.3439
700	0.6045	0.6045	−0.5047	−0.5047
800	0.4261	0.4261	−0.5904	−0.5904
900	0.2284	0.2284	−0.5801	−0.5801
1000	0.0599	0.0599	−0.4762	−0.4762

Table 2 The response of the sampling period ($T=0.5s$)

Time step	Matlab (x_1)	Taylor (x_1) ($N=1$)	Taylor (x_1) ($N=3$)	Taylor (x_1) ($N=7$)
4	1.1224	1.1088	1.1228	1.1224
8	0.6667	0.6283	0.6673	0.6666
12	0.3968	0.4515	0.3956	0.3968
16	0.6995	0.8317	0.6969	0.6994
20	1.1410	1.2242	1.1396	1.1410
24	1.0890	1.0666	1.0894	1.0889
28	0.6180	0.5848	0.6183	0.6177
32	0.4028	0.4689	0.4012	0.4026
36	0.7519	0.8851	0.7491	0.7517
40	1.1650	1.2370	1.1636	1.1649
Time step	Matlab (x_2)	Taylor (x_2) ($N=1$)	Taylor (x_2) ($N=3$)	Taylor (x_2) ($N=7$)
4	0.8112	0.9600	0.8084	0.8112
8	0.9578	1.0023	0.9570	0.9578
12	0.5623	0.5535	0.5624	0.5623
16	0.1914	0.2566	0.1900	0.1914
20	0.3764	0.5370	0.3732	0.3764
24	0.8523	0.9929	0.8497	0.8523
28	0.9373	0.9712	0.9367	0.9373
32	0.5096	0.5029	0.5096	0.5096
36	0.1820	0.2590	0.1803	0.1819
40	0.4238	0.5889	0.4205	0.4238

$[kT+\gamma_1, kT+T)$. We choose $N=3$ and $l=1$, $m=0$, $n=1$; $l=4$, $m=3$, $n=4$ and $l=8$, $m=6$,

Table 3 The response of the sampling period ($T=0.5s$) ⟨using scaling and squaring method⟩

Time step	Matlab (x_1)	Taylor (x_1) $l=1$, $m=0$, $n=1$	Taylor (x_1) $l=4$, $m=3$, $n=4$	Taylor (x_1) $l=8$, $m=6$, $n=8$
4	1.1224	1.1225	1.1224	1.1224
8	0.6667	0.6668	0.6666	0.6666
12	0.3968	0.3966	0.3968	0.3968
16	0.6995	0.6991	0.6994	0.6995
20	1.1410	1.1408	1.1410	1.1410
24	1.0890	1.0890	1.0889	1.0889
28	0.6180	0.6178	0.6177	0.6177
32	0.4028	0.4024	0.4026	0.4026
36	0.7519	0.7513	0.7517	0.7517
40	1.1650	1.1647	1.1649	1.1649
Time step	Matlab (x_2)	Taylor (x_2) $l=1$, $m=0$, $n=1$	Taylor (x_2) $l=4$, $m=3$, $n=4$	Taylor (x_2) $l=8$, $m=6$, $n=8$
4	0.8112	0.8107	0.8112	0.8112
8	0.9578	0.9577	0.9578	0.9578
12	0.5623	0.5623	0.5623	0.5623
16	0.1914	0.1912	0.1914	0.1913
20	0.3764	0.3759	0.3764	0.3764
24	0.8523	0.8519	0.8523	0.8523
28	0.9373	0.9372	0.9373	0.9373
32	0.5096	0.5096	0.5096	0.5095
36	0.1820	0.1817	0.1819	0.1819
40	0.4238	0.4233	0.4238	0.4238

$n=8$ respectively. The numerical differences between the Matlab ODE solver and the proposed method for state x_1 and x_2 are shown in Table 3. In the case of $l=1$, $m=0$, $n=1$ the numerical differences between the Matlab ODE solver and the proposed method for state x_1 range from -1.277×10^{-4} to 5.967×10^{-4} and those for state x_2 range from -0.231×10^{-4} to 5.250×10^{-4}; in the case of $l=4$, $m=3$, $n=4$ the numerical differences between the Matlab ODE solver and the proposed method for x_1 staterange from -0.330×10^{-4} to 2.529×10^{-4} and those for state x_2 range from -4.040×10^{-5} to 5.067×10^{-5}; and in the case of $l=8$, $m=6$, $n=8$ the numerical differences between the Matlab ODE solver and the proposed method for state x_1 range from -0.411×10^{-4} to 2.767×10^{-4} and those for state

Table 4 The computing time data for various (l, m, n, N) $(T=0.5s)$

Extrapolation to the limit parameters (l, m, n)	Order of the Taylor method N	computing time (s)
(1, 0, 1)	3	7.18
(4, 3, 4)	3	21.21
(8, 6, 8)	3	386.85
Single step Taylor method	7	7.56

Table 5 The computing time data for various (l, m, n, N) $(T=0.5s)$

Extrapolation to the limit parameters (l, m, m)	Order of the Taylor method N	computing time (s)
(0, 0, 0)	25	8.31
(1, 0, 1)	10	4.89
(2, 1, 2)	7	8.14
(3, 2, 3)	4	10.24
(4, 3, 4)	3	16.27

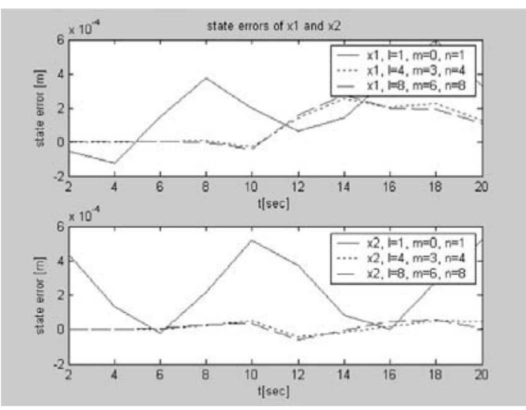

Fig. 3 State error responses of the simplified automobile for the case 2

x_2 range from -5.671×10^{-5} to 5.490×10^{-5}. The differences in the responses of the Taylor method and the Matlab solver are shown in Fig. 3. The computing time for these three cases and single step Taylor method are shown in Table 4. It is obvious that the results of $(N=3,\ l=1,\ m=0,\ n=1)$ case are adequate enough. Because the sampling period is not very large the computing time between single step Taylor method and Taylor method with SST does not present a significant difference. In the following case when the sampling time and time delay is larger the advantage of using SST is more obvious. From these data we can conclude that SST is good for the nonlinear case and after using this technique one can use only a small N to ensure the desired accurate results, which is more apparent in the case where the sampling interval T is very large. It would be prudent to choose the Extrapolation to the limit parameters (l, m, n). When some small parameters (l, m, n) can satisfy the desired accuracy it is not useful to choose larger ones because it will aggravate the computing task significantly. Then we continue to enlarge the sampling interval. We can get a series of computing times as the Extrapolation to the limit parameters (l, m, m) change, which are shown in the Table 5 $(T=5,\ D_1=2,\ D_2=8)$. From Table 5 we can conclude that in the case where the sampling period T is large enough it is more efficient to use the SST instead of single step Taylor-Lie method. When $(T=20,\ D_1=12,\ D_2=28)$ the computing time is calculated by Extrapolation to the limit parameters (l, m, n) and change is shown in Table 6. The differences in the responses the Taylor method and the Matlab solver between the case of single step Taylor method $(N=160)$ and the case of using the SST $(N=70,\ l=1,\ m=0,\ n=1)$ and $(N=20,\ l=2,\ m=1,\ n=2)$ are shown in Fig. 4. As the sampling period becomes larger and larger we have to enlarge N to

Table 6 The computing time data for various (l, m, n, N) $(T=20s)$

Extrapolation to the limit parameters (l, m, n)	Order of the Taylor method N	computing time (s)
(0, 0, 0)	160	156.15
(1, 0, 1)	70	88.28
(2, 1, 2)	20	71.04

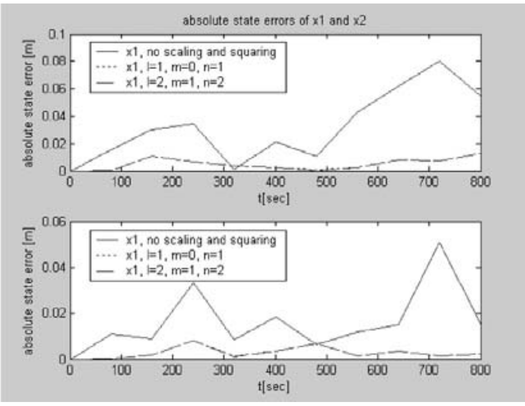

Fig. 4 State error responses of the simplified automobile for the case 3

satisfy the desired accuracy which will aggravate the computing pressure highly if we do not use the SST. When T is large enough, then $A^{[l]}T^{l}/l!$ might become extremely large (due to the finite-precision arithmetic) before it becomes small at higher powers, where convergence takes over. Consequently, in this case when the sampling time and delay time are very large we can not obtain accurate results using only the single-step Taylor method, especially in the case of the system whose fastest time constant is small. This makes it very difficult to control the system and ensure its stability. The SST can overcome this problem.

6. Conclusions

A new approach for discrete-time representation of nonlinear control systems with delayed multi-input in control is proposed. This approach is based on the ZOH assumption and the Taylor-series expansion which is obtained as a solution of a continuous-time system. The effect of sampling on system-theoretic properties of nonlinear systems with time-delayed multi-input, such as equilibrium properties and stability is examined. The well-known SST is expanded to the nonlinear case when the sampling period is too large.

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